Macroeconomics II: Optimal Taxation

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Economy

- I will start with a more general version so that you can adapt the problem to different specifications.
- However, from the very beginning I will focus on the case where the production has a unique input (labor) and government income is limited to linear taxes on labor income and trade of state dependent bonds.
- If you are interested in this topic (including some of the proofs) I suggest reading Chari and Kehoe's "Optimal Fiscal and Monetary Policy" in *Handbooks in Economics* (1999).

Households

- Let's start with the households. As seen in class, the economy we have in mind is one of a representative agent that solves (16.9.1)

$$\max_{\{c_t(s^t),l_t(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi_t \left(s^t\right) \beta^t u \left(c_t \left(s^t\right), l_t \left(s^t\right)\right)$$

subject to a sequence of budget constraints of the form

$$c_t + \sum_{s_{t+1}} P_t \left(s_{t+1}|s^t\right) b_{t+1} \left(s_{t+1}|s^t\right) \leq \left(1 - \tau_t^n \left(s^t\right)\right) w_t \left(s^t\right) + b_t \left(s^t\right)$$

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and a No Ponzi condition on the limit (for each possible infinite history s^{∞})

$$\lim_{T\to\infty}\left(\prod_{i=0}^{\infty} R_i^{-1}\left(s_{i+1}|s^i\right)\right)\frac{b_{T+1}\left(s_{T+1}|s^T\right)}{R_T\left(s^T\right)}=0 \text{ for all } s^{\infty}$$

where $R_i^{-1}\left(s_{i+1}|s^i\right)=P_i\left(s_{i+1}|s^i\right)$. (*Question:* How would the budget constraint look if we only had risk free bonds?)
Iterating forward on the budget constraint and applying the (many) No Ponzi constraints we get the time 0 budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right)c_t\left(s^t\right)\leq\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right)\left[1-\tau_t^n\left(s^t\right)\right]w_t\left(s^t\right)n_t\left(s^t\right)+b_0$$

where

$$\begin{aligned} q_{t+1}^0\left(s^{t+1}\right) &= P\left(s_{t+1}|s^t\right)q_t^0\left(s^t\right). \\ q_0^0\left(s^t\right) &= 1. \end{aligned}$$

(Note that given this notation, the No Ponzi condition can be rewritten as $\lim_{T\to\infty} q_{t+1}\left(s_{t+1}|s^t\right)b_{T+1}\left(s_{T+1}|s^T\right)=0$. The book imposes a less restrictive one, which might be sufficent but I think it allows for diverging assets as long as they do so at the same pace...)
Also recall that agents have a time limit to distribute between leisure and labor, which we simplify to one unit, i.e.

$$l_t\left(s^t\right)+n_t\left(s^t\right)=1.$$

- Hence, the HH problem can be summarized by

$$\max_{\{c_t(s^t),l_t(s^t)\}}\sum_{t=0}^{\infty}\sum_{s^t}\pi_t\left(s^t\right)\beta^t u\left(c_t\left(s^t\right),1-n_t\left(s^t\right)\right)$$

$$s.t.$$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right)c_t\left(s^t\right)\leq\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right)\left[1-\tau_t^n\left(s^t\right)\right]w_t\left(s^t\right)n_t\left(s^t\right)+b_0$$

$$b_0 \text{ given.}$$

- Note that if we had capital we would need to use a no-arbitrage arguement as well to simplify the time-0 budget constraint.

- *Question*: How would this change if we changed the taxes the government can impose? What if we had taxes on consumption for example?

Firms

- The firm's problem (just like in previous sections of this course) is a static one where they decide each period how much labor to hire , so as to maximize profits.

$$\max_{\{n_t(s^t)\}} F\left(n_t\left(s^t\right)\right) - w_t\left(s^t\right) n_t\left(s^t\right)$$

  where we use the fact that wages are indexed by the price of the good.

- Note that I am abstracting from the case where the state also affects the productivity of the firm, i.e. $F\left(n_t\left(s^t\right), s^t\right)$!

The Ramsey Problem

- Let's define some elements for this economy.

  1. A *feasible allocation* is a sequence $\left(c_t\left(s^t\right), n_t\left(s^t\right), g_t\left(s^t\right)\right)$ that satisfies the resource constraint

  $$c_t\left(s^t\right) + g_t\left(s^t\right) = F\left(n_t\left(s^t\right)\right).$$

  2. A *price system* is a non-negative sequence $\left(w_t\left(s^t\right), q_t\left(s^t\right)\right)$ (We could also define the returns on bonds from this).
  3. A *government policy* is a sequence $\left(g_t\left(s^t\right), \tau_t^n\left(s^t\right), b_{t+1}\left(s_{t+1}|s^t\right)\right)$.
  4. Given b_0, a *competitive equilibrium* is a feasible allocation, a price system and a government policy such that:
     - Given the price system and government policy, the allocation $\left(c_t\left(s^t\right), n_t\left(s^t\right)\right)$ solves the HH problem.
     - Given the allocation $\left(c_t\left(s^t\right), n_t\left(s^t\right)\right)$ and the price system, the govenrment policy satisfies the government budget constraint each period

     $$g_t\left(s^t\right) = \tau_t^n\left(s^t\right) w_t\left(s^t\right) n_t\left(s^t\right) + \sum_{s_{t+1}} \frac{q_{t+1}^0\left(s_{t+1}|s^t\right)}{q_t^0\left(s^t\right)} b_{t+1}\left(s_{t+1}|s^t\right) - b_t\left(s^t\right)$$

     (Note that this defines the path of bonds. *Question*: How would this change if the government had other sources of income?).
  5. Given b_0, the Ramsey problem is to choose a competitive equilibrium that maximizes the household utility.

- Note that there are multiple competitive equilibria, indexed by different government policies (even if we take the sequence of government expenses as given, there are many ways to finance them!). Hence, this multiplicity is what motivates the Ramsey problem. What is the best way to finance the government given that it is restricted to operate in a competitive equilibria?

- In order to solve this problem we will follow what is known as the primal approach, which consists of 4 steps:

  1. Obtains FOCs from household's and firm's (and no arbitrage conditions if we had any), to solve for prices and taxes in terms of exclusively allocations.
  2. Substitute these expressions for taxes and prices in the household's present-value budget constraint. This way we get an intertemporal constraint involving only the allocation. Let this be the "Implementability Condition" since it guarantees both the HH budget constraint and the allocation being obtained by some Competitive Equilibrium prices.
  3. Solve for the Ramsey allocation by maximizing expected utility subject to the implementability condition and the resource constraint (Note this two imply the goverment budget constraint).
  4. Use the allocations to back out prices and taxes from the FOCs of step one, and bonds from the government budget constraint for example (not the present value one).

Step 1: HH's and Firm's problem

Households

- Let λ be the Lagrange multiplier on the HH's present value budget constraint.
- To simplify notation let

$$\begin{aligned} u_c\left(s^t\right) &= u_1\left(c_t\left(s^t\right), 1 - n_t\left(s^t\right)\right) \\ u_l\left(s^t\right) &= u_2\left(c_t\left(s^t\right), 1 - n_t\left(s^t\right)\right) \end{aligned}$$

- The FOC with respect to consumption is

$$\pi_t\left(s^t\right)\beta^t u_c\left(s^t\right) = \lambda q_t^0\left(s^t\right)$$

  and with respect to labor

$$\pi_t\left(s^t\right)\beta^t u_l\left(s^t\right) = \lambda q_t^0\left(s^t\right)\left[1 - \tau_t^n\left(s^t\right)\right] w_t\left(s^t\right).$$

- By combining different periods of the one of consumption we get

$$q_t^0\left(s^t\right) = \beta^t \pi_t\left(s^t\right)\frac{u_c\left(s^t\right)}{u_c\left(s_0\right)}.$$

  And combining the FOC of consumption and of labour for the same period and state,

$$\frac{u_l\left(s^t\right)}{u_c\left(s^t\right)} = \left[1 - \tau_t^n\left(s^t\right)\right] w_t\left(s^t\right).$$

Firm

- From the firm's problem, we obtain that in a competitive equilibrium with positive production it must be that
$$F_n\left(s^t\right) = w_t\left(s^t\right).$$
Here we use the usual argument of bounded resources (labor in this case).

Step 2: Obtain Implementability Condition

- Now we replace the results from step 1 in the HH's budget constraint to obtain the Implementability Condition (IC)
- From before we had that
$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t\left(s^t\right) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right)\left[1-\tau_t^n\left(s^t\right)\right] w_t\left(s^t\right) n_t\left(s^t\right) + b_0.$$
- Given the utility function we know that it will be satisfied with equality, i.e.
$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right)\left[1-\tau_t^n\left(s^t\right)\right] w_t\left(s^t\right) n_t\left(s^t\right) + b_0.$$
- The IC condition refers to the fact that HH's must be exhausting their income completely, and hence the Ramsey planner is not free to choose household expenditures strictly less than their incomes. This causes the change in sign which will be important for the sign of its Lagrange multiplier. Replacing we obtain
$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right)\frac{u_c\left(s^t\right)}{u_c\left(s_0\right)} c_t\left(s^t\right) \geq \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right)\frac{u_c\left(s^t\right)}{u_c\left(s_0\right)}\frac{u_l\left(s^t\right)}{u_c\left(s^t\right)} n_t\left(s^t\right) + b_0$$
and reorganizing
$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right)\left[u_c\left(s^t\right) c_t\left(s^t\right) - u_l\left(s^t\right) n_t\left(s^t\right)\right] \geq u_c\left(s_0\right) b_0.$$
This is the IC we were after. The (RC) and the government's budget constraint end up implying that this must hold with equality, so
$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right)\left[u_c\left(s^t\right) c_t\left(s^t\right) - u_l\left(s^t\right) n_t\left(s^t\right)\right] = u_c\left(s_0\right) b_0.$$
But keep in mind the direction of the inequality for the sign of Φ later on. (*Question:* How would this be affected by previous question on just trading risk free assets? This is a lot more complicated since another restriction is added at every state...If you are interested it is in RMT4)

Step 3: Ramsey Problem

- We want to maximize the expected utility subject to (IC) and the resource constraint in each period (RC). Let Φ be the lagrange multiplier on the (IC) and $\beta^t \pi_t(s^t)\theta_t(s^t)$ the one on the (RC) at time t and state s^t.
- For ease of notation it always useful to define a term that summarizes the utility function taking into account the distortion generated by the (IC), i.e.

  $$V\left[c_t\left(s^t\right), n_t\left(s^t\right), \Phi\right] = u\left(c_t\left(s^t\right), 1 - n_t\left(s^t\right)\right) + \Phi\left[u_c\left(s^t\right) c_t\left(s^t\right) - u_l\left(s^t\right) n_t\left(s^t\right)\right].$$

- Then, the Lagrangian of the Ramsey Problem is

  $$J = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right)\left\{V\left[c_t\left(s^t\right), n_t\left(s^t\right), \Phi\right] + \theta_t\left(s^t\right)\left[F\left(n_t\left(s^t\right)\right) - c_t\left(s^t\right) - g_t\left(s^t\right)\right]\right\} - \Phi u_c\left(s_0\right) b_0$$

  which will be maximized with respect to $c_t(s^t), n_t(s^t)$.
- As we know from class the probem at time 0 is different from the one at time 1 (which leads to the time inconsistency). Hence we need to divide the FOCs in 2 parts: a) $t \geq 1$ and b) $t = 0$.

$t \geq 1$

- The FOCs of this problem with respect to consumption is

  $$V_c\left(s^t\right) = \theta_t\left(s^t\right)$$

  and with respect to labor

  $$V_n\left(s^t\right) = -\theta_t\left(s^t\right) F_n\left(s^t\right).$$

- Here V_c and V_n hide a lot of elements from the distorted objective function, i.e.

  $$V_c\left(s^t\right) = u_c\left(s^t\right) + \Phi\left[u_{cc}\left(s^t\right) c_t\left(s^t\right) + u_c\left(s^t\right) - u_{lc}\left(s^t\right) n_t\left(s^t\right)\right]$$

  and

  $$V_n\left(s^t\right) = -u_l\left(s^t\right) + \Phi\left[-u_{cl}\left(s^t\right) c_t\left(s^t\right) + u_{ll}\left(s^t\right) n_t\left(s^t\right) - u_l\left(s^t\right)\right].$$

$t = 0$

- The FOCs of this problem with respect to consumption is

  $$V_c\left(s_0\right) = \theta_0\left(s_0\right) + \Phi u_{cc}\left(s_0\right) b_0$$

  and with respect to labor

  $$V_n\left(s_0\right) = -\theta_0\left(s_0\right) F_n\left(s_0\right) - \Phi u_{cl}\left(s_0\right) b_0.$$

Solution

- Then, the solution to the Ramsey problem is given by the FOCs just found, the (IC) and the series of (RC).
- We can write them all together here for ease of work later on:

$$\begin{aligned}
u_c\left(s^t\right)(1+\Phi)+\Phi\left[u_{cc}\left(s^t\right)c_t\left(s^t\right)-u_{lc}\left(s^t\right)n_t\left(s^t\right)\right]-\theta_t\left(s^t\right) &= 0 \text{ for } t\geq 1\\
-u_l\left(s^t\right)(1+\Phi)-\Phi\left[u_{cl}\left(s^t\right)c_t\left(s^t\right)-u_{ll}\left(s^t\right)n_t\left(s^t\right)\right]+\theta_t\left(s^t\right)F_n\left(s^t\right) &= 0 \text{ for } t\geq 1\\
u_c(s_0)(1+\Phi)+\Phi\left[u_{cc}(s_0)c_0(s_0)-u_{lc}(s_0)n_t(s_0)\right]-\theta_0(s_0)-\Phi u_{cc}(s_0)b_0 &= 0\\
-u_l(s_0)(1+\Phi)-\Phi\left[u_{cl}(s_0)c_0(s_0)-u_{ll}(s_0)n_t(s_0)\right]+\theta_0(s_0)F_n(s_0)-\Phi u_{cl}(s_0)b_0 &= 0
\end{aligned}$$

and

$$\begin{aligned}
F\left(n_t\left(s^t\right)\right)-c_t\left(s^t\right)-g_t\left(s^t\right) &= 0 \text{ for all } t \text{ and } s^t\\
\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left[u_c\left(s^t\right)c_t\left(s^t\right)-u_l\left(s^t\right)n_t\left(s^t\right)\right] &= u_c(s_0)b_0.
\end{aligned}$$

- Let the solution be $\{c_t^*(s^t), n_t^*(s^t)\}$. (Note that this solution need not be unique. For example we could have Laffer curve type of issues).

Step 4: Recover prices and taxes

- Having obtained the optimal allocation $\{c_t^*(s^t), n_t^*(s^t)\}$ we now want to find the prices and taxes that implement this allocation as a competitive equilibrium. How do we know there exist such prices and policies? Because we had it as a requirement for it to be a solution (Recall the Implementability condition).
- To obtain this prices and taxes we use the FOCs from step one.
  - From the Firm's problem,
  $$w_t^*\left(s^t\right)=F_n\left(n_t^*\left(s^t\right)\right).$$
  - From the Household's problem i.e.
  $$q_t^{0*}\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u_c\left(c_t^*\left(s^t\right),1-n_t^*\left(s^t\right)\right)}{u_c\left(c_0^*\left(s_0\right),1-n_0^*\left(s_0\right)\right)}$$
  and
  $$\left[1-\tau_t^{n*}\left(s^t\right)\right]=\frac{u_l\left(c_t^*\left(s^t\right),1-n_t^*\left(s^t\right)\right)}{u_c\left(c_t^*\left(s^t\right),1-n_t^*\left(s^t\right)\right)}\frac{1}{w_t^*\left(s^t\right)}.$$
- Finally, if we want to obtain the trade of state contingent bonds we can use the government's budget constraint for example,

$$g_t\left(s^t\right)=\tau_t^n\left(s^t\right)w_t\left(s^t\right)n_t\left(s^t\right)+\sum_{s_{t+1}}\frac{q_{t+1}^0\left(s_{t+1}|s^t\right)}{q_t^0\left(s^t\right)}b_{t+1}\left(s_{t+1}|s^t\right)-b_t\left(s^t\right)$$

A Simplifying Example

Utility Function

- Suppose that preferences are separable such that
$$u(c, 1-n) = v(c) + h(1-n).$$
- Moreover, assume that the utility of consumption is linear, i.e.
$$v(c) = c$$
and the utility of leisure is quadratic but guaranteeing that it always has positive marginal utility ($h' \geq 0$ for $n \in [0,1]$)) and strictly concave ($h'' < 0$ for all $n \in [0,1]$) , i.e.
$$h(1-n) = (1-n) - \alpha(1-n)^2$$
so
$$\begin{aligned} h'(1-n) &= 1 - 2\alpha(1-n) \\ &= -2\alpha + 2\alpha n \\ h''(1-n) &= -2\alpha \end{aligned}$$
so we need that $\alpha \in \left(0, \frac{1}{2}\right)$.
- Note that these assumptions imply that first derivatives are
$$\begin{aligned} u_c\left(s^t\right) &= 1 \\ u_l\left(s^t\right) &= 1 - 2\alpha\left(1 - n_t\left(s^t\right)\right) \end{aligned}$$
and second derivatives are
$$\begin{aligned} u_{cc}\left(s^t\right) &= 0 \\ u_{ll}\left(s^t\right) &= -2\alpha \\ u_{cl}\left(s^t\right) &= 0 \\ u_{lc}\left(s^t\right) &= 0. \end{aligned}$$
- This will simplify our previously found conditions for the Ramsey planner enormously.

Production Function

- Also, assume the production function is linear, i.e.
$$F(n) = n,$$
so $F'(n) = 1$.
- Note this will make the wage equal to one in all periods and states, i.e.
$$w_t\left(s^t\right) = F_n\left(n_t\left(s^t\right)\right) = 1.$$

Markov Process

- Finally we assume that the state is Markov, i.e. $\pi_{t+1}(s_{t+1}|s^t) = \pi(s_{t+1}|s_t)$.
- Also, the government expenses depend only on the current state, i.e. $g_t(s^t) = g(s_t)$.
- Moreover, I will assume there are only two states $s_t \in [s^-, s^+]$ and $g(s^-) = g^- < g^+ = g(s^+)$, assuming $g_0 = g^-$ so we start with a small government. And let $\pi(s^+|s^+) = \pi_+$ and $\pi(s^-|s^-) = \pi_-$.

Ramsey Problem

- Now I take the first two steps of the solution as given, and jump directly to the Solutions section of Step 3 above. Here I replace the simplifications made before to obtain

$$\begin{aligned} 1+\Phi-\theta_t\left(s^t\right) &= 0 \text{ for } t \geq 1 \\ -\left[1-2\alpha+2\alpha n_t\left(s^t\right)\right](1+\Phi)-\Phi 2\alpha n_t\left(s^t\right)+\theta_t\left(s^t\right) &= 0 \text{ for } t \geq 1 \\ 1+\Phi-\theta_0\left(s_0\right) &= 0 \\ -\left[1-2\alpha+2\alpha n_0\left(s_0\right)\right](1+\Phi)-\Phi 2\alpha n_0\left(s_0\right)+\theta_0\left(s_0\right) &= 0 \end{aligned}$$

  and

$$\begin{aligned} n_t\left(s^t\right)-c_t\left(s^t\right)-g_t\left(s_t\right) &= 0 \text{ for all } t \text{ and } s^t \\ \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left[c_t\left(s^t\right)-(1-2\alpha)n_t\left(s^t\right)-2\alpha n_t^2\left(s^t\right)\right] &= b_0. \end{aligned}$$

- First, note that time 0 FOCs and $t \geq 1$ FOCs are equal now, due to the separability of the utility function and the linearity of consumption.
- From the first equation we see that $\theta_t(s^t)$ should be constant for all t and state, i.e.

$$\theta_t\left(s^t\right) = 1+\Phi \text{ for all } t \text{ and } s^t.$$

- Then, going to the other FOC

$$\begin{aligned} n_t\left(s^t\right)\left(1+\frac{\Phi}{1+\Phi}\right) &= 1 \\ n_t\left(s^t\right) &= \frac{1+\Phi}{1+2\Phi} \end{aligned}$$

  where we note that labor supply will be constant, independent of the state (i.e. the government expenses). Hence, production will be constant and consumption adjustments will accomodate the changes in government expenses.

- Using the Resource constraint we obtain consumption

$$c_t\left(s^t\right)=\frac{1+\Phi}{1+2\Phi}-g_t\left(s_t\right)\text{ for all } t \text{ and } s^t.$$

- Finally, using the Implementability constraint we can obtain Φ

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left\{\frac{1+\Phi}{1+2\Phi}-g_t\left(s_t\right)-(1-2\alpha)\frac{1+\Phi}{1+2\Phi}-2\alpha\left[\frac{1+\Phi}{1+2\Phi}\right]^2\right\}=b_0$$

from where we can separate the non-stochastic terms from the stochastic ones

$$\frac{1+\Phi}{1+2\Phi}\frac{2\alpha}{1-\beta}\left[1-\frac{1+\Phi}{1+2\Phi}\right]=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)g_t\left(s_t\right)+b_0$$

To obtain $\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)g_t\left(s_t\right)$, we can define the present value of expenses as G^- when $s_0=s^-$ and G^+ if $s_0=s^+$, and find G recursively

$$\begin{bmatrix}G^-\\G^+\end{bmatrix}=\begin{bmatrix}g^-\\g^+\end{bmatrix}+\beta\underbrace{\begin{bmatrix}\pi_- & 1-\pi_-\\1-\pi_+ & \pi_+\end{bmatrix}}_{\Pi}\begin{bmatrix}G^-\\G^+\end{bmatrix}$$

from which we obtain (under invertibility assumptions)

$$\begin{bmatrix}G^-\\G^+\end{bmatrix}=(I-\beta\Pi)^{-1}\begin{bmatrix}g^-\\g^+\end{bmatrix}.$$

So we get that, assuming we start with $G_0\in\{G^-,G^+\}$

$$\frac{1+\Phi}{1+2\Phi}\frac{2\alpha}{1-\beta}\left[\frac{\Phi}{1+2\Phi}\right]=G_0+b_0$$

where we can see have a laffer curve typer of result. There will be two Φ that solve this (i.e. work little with high taxes or work more with higher taxes). Intuitively, we want the one with the lowest (positive) Φ because it is closest to the optimal (i.e. $\Phi=0$).

$$\Phi^2\underbrace{\left[\frac{2\alpha}{1-\beta}-4G_0-4b_0\right]}_{=K}+\Phi\underbrace{\left[\frac{2\alpha}{1-\beta}-4G_0-4b_0\right]}_{=K}-(G_0+b_0)=0$$

so (because one of them will be negative)

$$\Phi^*=\max\left\{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\frac{(G_0+b_0)}{K}},0\right\}.$$

Note that if $-b_0=G_0$, i.e. the government has enough funds that its whole present value cost of expenses, $\Phi=0$ and we have no distortions.

- Then this defines all the allocations.
- To obtain prices and taxes we use the FOCs from step 1:

$$q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)$$

And

$$\begin{aligned}1-\tau_t^n\left(s^t\right) &= 1-2\alpha\left(1-n_t\left(s^t\right)\right)\\ \tau_t^n\left(s^t\right) &= 2\alpha\frac{\Phi^*}{1+2\Phi^*}\end{aligned}$$

which is constant in time. Recall that if $\Phi^*=0$, we said there would be no distortions, which is seen here as $\tau_t^n\left(s^t\right)=0$ for all t and s^t. Since it is constant in time, we expect the government to save money when it has low expenses and borrow more when it has high expenses. Regarding bonds, we use the government budget constraint for example

$$\begin{aligned}g_t\left(s_t\right) &= \tau_t^n\left(s^t\right)n_t\left(s^t\right)+\sum_{s_{t+1}}\frac{q_{t+1}^0\left(s_{t+1}|s^t\right)}{q_t^0\left(s^t\right)}b_{t+1}\left(s_{t+1}|s^t\right)-b_t\left(s^t\right)\\ &= 2\alpha\frac{\Phi^*}{1+2\Phi^*}\frac{1+\Phi^*}{1+2\Phi^*}+\sum_{s_{t+1}}\beta\pi_t\left(s_{t+1}|s_t\right)b_{t+1}\left(s_{t+1}|s^t\right)-b_t\left(s^t\right)\end{aligned}$$

so we can once again stack it in vector form, for $t\geq 1$ (Note the tax income is constant and expenses vary, so the difference must come from the bonds)

$$\begin{aligned}\begin{bmatrix}b^-\\ b^+\end{bmatrix} &= 2\alpha\frac{\Phi^*}{1+2\Phi^*}\frac{1+\Phi^*}{1+2\Phi^*}-\begin{bmatrix}g^-\\ g^+\end{bmatrix}+\beta\underbrace{\begin{bmatrix}\pi_- & 1-\pi_-\\ 1-\pi_+ & \pi_+\end{bmatrix}}_{\Pi}\begin{bmatrix}b^-\\ b^+\end{bmatrix}\\ &= \left(I-\beta\Pi\right)^{-1}\begin{bmatrix}\tau n-g^-\\ \tau n-g^+\end{bmatrix}\\ &= \left(1-\beta\right)^{-1}\tau n-\begin{bmatrix}G^-\\ G^+\end{bmatrix}\end{aligned}$$

from where we should obtain that $b^->b^+$ so the government would promise to pay more when it has low expenses (i.e. it insures itself in the high expense assets by trading assets). Or we can see this by iterating and replacing $b_{t+1}\left(s_{t+1}|s_t\right)$, using the transversality condition

$$
\begin{aligned}
b_t\left(s^t\right) &= 2\alpha\frac{\Phi^*}{1+2\Phi^*}\frac{1+\Phi^*}{1+2\Phi^*} - g_t\left(s_t\right) + \\
&\quad + \sum_{s_{t+1}} \beta\pi_t\left(s_{t+1}|s_t\right)\left[\begin{array}{c} 2\alpha\frac{\Phi^*}{1+2\Phi^*}\frac{1+\Phi^*}{1+2\Phi^*} - g_{t+1}\left(s_{t+1}\right) \\ +\sum_{s_{t+2}}\beta\pi_t\left(s_{t+2}|s_{t+1}\right)b_{t+2}\left(s_{t+2}|s^{t+1}\right)\end{array}\right] \\
&= 2\alpha\frac{\Phi^*}{1+2\Phi^*}\frac{1+\Phi^*}{1+2\Phi^*}\left(1+\beta\right) - g_t\left(s_t\right) - \sum_{s_{t+1}}\beta\pi_t\left(s_{t+1}|s_t\right)g_{t+1}\left(s_{t+1}\right) + \\
&\quad + \sum_{s_{t+1}}\sum_{s_{t+2}}\beta^2\pi_t\left(s_{t+2}|s_t\right)b_{t+2}\left(s_{t+2}|s_{t+1}\right) \\
&= \ldots \\
&= \frac{2\alpha}{1-\beta}\frac{\Phi^*}{1+2\Phi^*}\frac{1+\Phi^*}{1+2\Phi^*} - G_t \\
&= \left(1-\beta\right)^{-1}\tau n - G_t.
\end{aligned}
$$

The reason why debt jumps to the new steady state is because we have complete markets. In an incomplete market economy assets (e.g. a unique risk-free asset) would be used to smooth the distortion from taxes. Here, state contingent debt (a.k.a. Arrow securities) is the best way to do this smoothing!

- Why does the government do this? The government issues debt that pays in low-expense periods and acquires assets that pay when it has lots of expenses. If it didn't do this (higher taxes than needed when in low expenses to buy assets that pay in bad times) it would be forced to have higher taxes in high expense periods (s^+), which would in turn increase the spread in labor supply (and consumption as well). Since taxes distort the economy, the government wishes to minimize the distortion by spreading the tax burden across states.